MATH 2010B Advanced Calculus I Lecture Notes Week 12

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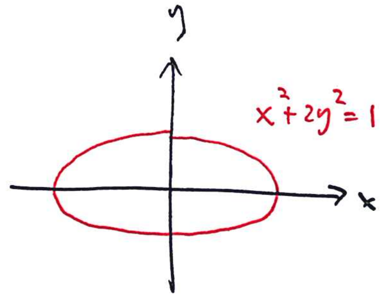
Last time ... 2^{nd} Derivative test, Taylor's Theorem.

Constrained Optimization I



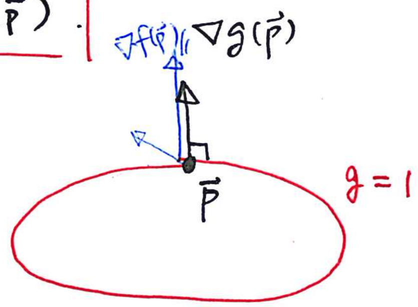
E.g 1: max/min $f(x,y) = xy$

under $\underbrace{g(x,y) = x^2+2y^2 = 1}_{\text{constraint}}$

Recall: Without constraint $\Rightarrow$ $\nabla f(\vec{p}) = \vec{0}$ $\vec{p}$: critical pts

With one constraint $\Rightarrow$ at an extremum,

$$\boxed{\nabla f(\vec{p}) \,/\!/\, \nabla g(\vec{p})}.$$

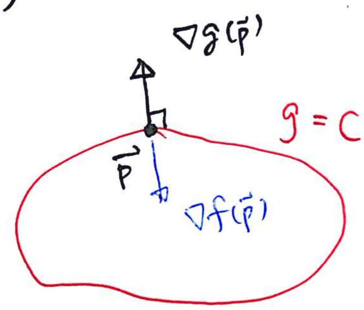


Remember: $\nabla f(\vec{p})$ is the "direction of fastest increase".

Theorem: (Lagrange Multiplier - 1 constraint)

Consider an optimization problem :

$$\begin{cases} \text{max/min} & f(x_1, \dots, x_n) \\ \text{under} & g(x_1, \dots, x_n) = c \end{cases}$$



If $\vec{p}$ is a local extremum for f s.t. $g(\vec{p}) = c$.

then when $\boxed{\nabla g(\vec{p}) \neq \vec{0}}$, then we have

$$\boxed{\nabla f(\vec{p}) = \underbrace{\lambda}_{\text{Lagrange multiplier.}} \nabla g(\vec{p})} \quad \text{for some } \lambda \in \mathbb{R}.$$

Remark: λ could be equal to 0.

Back to Eg. 1: By Lagrange multiplier:

At extremum $\vec{p}$, we have

$$(\clubsuit)\begin{cases} \nabla f(\vec{p}) = \lambda \nabla g(\vec{p}) & \leftarrow \text{2 equations} \\ g(\vec{p}) = 1 & \leftarrow \text{1 equation} \end{cases} \Bigg\} \text{ 3 equations!}$$

$\vec{p} = (x_0, y_0)$ λ 3 unknowns match!

By direct calculations,

$$\nabla f = (f_x, f_y) = (y, x)$$

$$\nabla g = (g_x, g_y) = (2x, 4y)$$

$$(\clubsuit) \Rightarrow \begin{cases} y = \lambda(2x) & \text{——— ①} \\ x = \lambda(4y) & \text{——— ②} \\ x^2 + 2y^2 = 1 & \text{——— ③} \end{cases}$$

Idea:

(A) Write x, y in terms of λ using ① & ②

(B) Put these into ③ to get an equation only in λ

(C) Solve for λ, find x, y using (A)

Sub ② into ①,

$$y = 2\lambda(4\lambda y) = 8\lambda^2 y.$$

Case 1: $y = 0 \overset{②}{\Rightarrow} x = 0 \Rightarrow$ violates ③.

$\nabla g = (2x, 4y) = \vec{0}$

$\Leftrightarrow x = y = 0$.

but $g(0,0) = 0 \neq 1$

not on the constraint curve.

Case 2: $y \neq 0 \Rightarrow 8\lambda^2 = 1 \Rightarrow \lambda = \pm \frac{1}{\sqrt{8}}$.

Plug into ①, $y = 2\left(\pm\frac{1}{\sqrt{8}}\right)x = \pm\frac{1}{\sqrt{2}}x$.

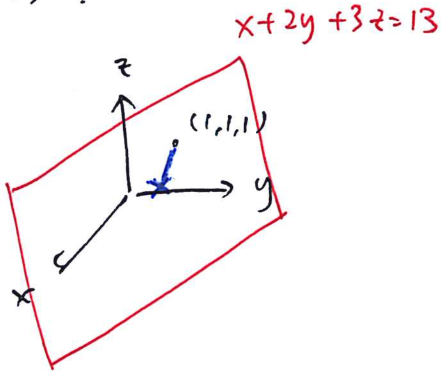
Plug into ③, $x^2 + 2\left(\pm\frac{1}{\sqrt{2}}x\right)^2 = 1$

$$\Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}.$$

Get 4 solutions: $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$.

$f = xy$: $\frac{1}{2\sqrt{2}}$ max, $-\frac{1}{2\sqrt{2}}$ min, $-\frac{1}{2\sqrt{2}}$ min, $\frac{1}{2\sqrt{2}}$ max

E.g. 2: Find the point on the plane $x+2y+3z=13$ which is closest to the point $(1,1,1)$.



Sol: Set up the problem:

$$(\ast)\quad \begin{cases} \min\ f(x,y,z) = \operatorname{dist}\big((x,y,z),(1,1,1)\big) \\ \qquad\qquad\quad = \sqrt{(x-1)^2+(y-1)^2+(z-1)^2}. \\ \text{Under the constraint } x+2y+3z=13. \end{cases}$$

$(\ast)$ is equivalent to

$$(\ast\ast)\quad \begin{cases} \min\ f(x,y,z) = (x-1)^2+(y-1)^2+(z-1)^2. \\ \text{under } g(x,y,z) = x+2y+3z = 13 \end{cases}$$

Remark, Ex: transform the constraint away

Lagrange multiplier method:

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 13 \end{cases} \Rightarrow \begin{cases} 2(x-1) = \lambda(1) & \text{—— ①} \\ 2(y-1) = \lambda(2) & \text{—— ②} \\ 2(z-1) = \lambda(3) & \text{—— ③} \\ x+2y+3z = 13. & \text{—— ④} \end{cases}$$

$$① - ③ \Rightarrow \begin{cases} x = \frac{\lambda}{2}+1 \\ y = \lambda+1 \\ z = \frac{3\lambda}{2}+1. \end{cases} \xRightarrow[④]{\text{Plug in}} \left(\frac{\lambda}{2}+1\right)+2(\lambda+1)+3\left(\frac{3\lambda}{2}+1\right)=13$$

$$\Downarrow$$

$$7\lambda+6=13$$

$$\Downarrow$$

$$\lambda = 1$$

put back

Get $(x,y,z) = \underbrace{\left(\frac{3}{2}, 2, \frac{5}{2}\right)}_{\text{this is the closest point to } (1,1,1).}$.